

INDEKS PELANGI-3 KUAT GRAF BAT

STRONG 3-RAINBOW INDEXES OF BAT GRAPHS

Suci Yefri Fadhilah^{1§}, Annisa Maula Zakiya², Widya Reza²

¹Universitas Islam Negeri Imam Bonjol Padang

[§]Correspondence: suciyeфри16@gmail.com

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Abstrak

Misalkan G adalah graf tak trivial, terhubung, dan berwarna tepi dengan orde $m \geq 3$, di mana sisi-sisi yang berdekatan diperbolehkan memiliki warna yang sama. Sebuah pohon T dalam graf G disebut pohon pelangi jika semua sisi pada T memiliki warna yang berbeda. Untuk suatu himpunan simpul $S \subseteq V(G)$, jarak Steiner $sd(S)$ didefinisikan sebagai ukuran minimum dari pohon dalam G yang memuat semua simpul dalam S . Misalkan k adalah bilangan bulat dengan $2 \leq k \leq m$. Pewarnaan tepi pada graf G disebut pewarnaan pelangi- k kuat jika untuk setiap himpunan $S \subseteq V(G)$ dengan $|S| = k$, terdapat pohon pelangi berukuran $sd(S)$ yang memuat S . Penelitian ini menggunakan metode analitik-teoretis dengan pendekatan konstruksi pewarnaan dan analisis struktur graf untuk menentukan nilai indeks pelangi-3 kuat pada graf bat. Hasil penelitian menunjukkan bahwa nilai indeks pelangi-3 kuat dari graf bat $Bat(n)$ untuk $n \geq 3$ adalah 2 untuk $n = 3$, $n - 3$, untuk $4 \leq n \leq 6$ atau $n = 8$ dan n , untuk 7 or $n \geq 9$.

Kata Kunci: graf bat, pewarnaan pelangi, pohon pelangi, indeks 3- Pelangi kuat

Abstract

Let G be a nontrivial connected edge-colored graph of order $m \geq 3$, where adjacent edges are allowed to have the same color. A tree T in G is called a rainbow tree if all edges in T have distinct colors. For a subset of vertices $S \subseteq V(G)$, the Steiner distance $sd(S)$ is defined as the minimum size of a tree in G that contains all vertices in S . Let k be an integer with $2 \leq k \leq m$. An edge-coloring of G is called a strong k -rainbow coloring if, for every subset $S \subseteq V(G)$ with $|S| = k$, there exists a rainbow tree of size $sd(S)$ containing S . This study employs a theoretical-analytical method using coloring construction and graph structural analysis to determine the strong 3-rainbow index of bat graphs. The results show that the strong 3-rainbow index of the bat graph $Bat(n)$ for $n \geq 3$ is given by 2 for $n = 3$, $n - 3$, for $4 \leq n \leq 6$ or $n = 8$ dan n , for 7 or $n \geq 9$.

Keywords: bat graphs, rainbow coloring, rainbow tree, strong k -rainbow index

1. Introduction

Graph theory is a fundamental branch of discrete mathematics that is widely used to model and analyze various types of networks, particularly communication networks. In this context, reliability, security, and efficiency are key aspects that must be considered. To support the development of secure and interference-free communication systems, various techniques in graph theory have been introduced, one of which is graph coloring.

Graph coloring, especially edge coloring, can be used to represent the allocation of resources such as communication channels, frequencies, or encryption keys. In its development, the concept of rainbow coloring was introduced, where edges of a graph are colored in such a way that certain paths or structures consist of edges with distinct colors. This concept was later extended to the k -rainbow index, which generalizes connectivity from pairs of vertices to subsets of vertices. In this setting, each subset of vertices must be connected by a rainbow tree that spans all vertices in the subset.

A further extension of this concept leads to the strong k -rainbow index, which imposes a stricter condition than the k -rainbow index. In this case, each subset of vertices must be connected not only by a rainbow tree but also by a minimum Steiner tree. Thus, the strong k -rainbow index represents the minimum number of colors required to ensure both uniqueness of edge colors and efficiency in terms of the size of the connecting structure.

In the specific case of the strong 3-rainbow index, the objective is to guarantee that every set of three vertices in a graph can be connected by a minimum rainbow tree. From an application perspective, vertices may represent network devices such as routers or servers, while edges represent communication links between them. Assigning distinct colors to edges can be interpreted as allocating different communication channels or encryption schemes, thereby reducing interference and enhancing security in multi-terminal communication systems. Moreover, the study of the strong k -rainbow index provides theoretical insights into how structural properties of graphs, such as the presence of cycles and connectivity levels, influence rainbow connectivity.

Numerous previous studies have investigated the k -rainbow index and the strong k -rainbow index for various classes of graphs, including path graphs, cycle graphs, and complete graphs. These studies indicate that the values of these indices are strongly influenced by structural characteristics of the graph, such as vertex degrees, the existence of cycles, and graph diameter. This highlights the importance of graph structure in determining the efficiency of rainbow coloring.

Based on the above discussion, this study focuses on a newly defined class of graphs known as bat graphs. In particular, this research aims to analyze the strong k -rainbow index associated with this class of graphs. The results are expected to contribute to the theoretical development of discrete mathematics, especially in the area of graph theory and graph coloring.

2. Some Concepts

Graph coloring is a fundamental concept in graph theory that plays an important role in combinatorial optimization and network modeling. In general, graph coloring refers to the assignment of colors to elements of a graph, such as vertices or edges, subject to certain constraints. In this study, the focus is on edge coloring, where colors are assigned to the edges of a graph. Formally, for a graph $G = (V(G), E(G))$, an edge coloring is defined as a mapping

$$c: E(G) \rightarrow \{1, 2, \dots, t\},$$

where t denotes the number of colors used. In the context of rainbow coloring, adjacent edges are allowed to share the same color unless otherwise restricted.

An important extension of edge coloring is the concept of rainbow coloring, which is closely related to graph connectivity. A path in an edge-colored graph is called a *rainbow path* if all its edges have distinct colors. Based on this concept, the rainbow connection number, denoted by $rc(G)$, is defined as the minimum number of colors required so that every pair of vertices in a connected graph is connected by at least one rainbow path.

This concept is further generalized into the k -rainbow index. Let G be a connected graph of order $n \geq 3$, and let k be an integer with $2 \leq k \leq n$. For a subset $S \subseteq V(G)$, a *rainbow S -tree* is a tree that contains all vertices in S and whose edges have distinct colors. An edge coloring of G is called a *k -rainbow coloring* if, for every subset S of k vertices, there exists a rainbow S -tree in G . The k -rainbow index, denoted by $rx_k(G)$, is the minimum number of colors required to achieve such a coloring.

In particular, when $k = 2$, the k -rainbow index coincides with the rainbow connection number, that is,

$$rx_2(G) = rc(G).$$

Moreover, for any connected graph G of order n , the following monotonicity property holds:

$$rx_2(G) \leq rx_3(G) \leq \dots \leq rx_n(G).$$

Another important concept in this study is the Steiner tree. For a subset $S \subseteq V(G)$, the *Steiner distance*, denoted by $sd(S)$, is defined as the minimum number of edges in a tree that connects all vertices in S . The maximum Steiner distance over all subsets of size k is called the k -Steiner diameter, denoted by $sdiam_k(G)$. These parameters satisfy the inequality:

$$k - 1 \leq sdiam_k(G) \leq rx_k(G) \leq n - 1.$$

A further development of the k -rainbow index is the strong k -rainbow index, denoted by $srx_k(G)$. In this context, a *rainbow Steiner tree* is a Steiner tree whose edges have distinct colors and whose size equals the Steiner distance $sd(S)$. An edge coloring of G is called a *strong k -rainbow coloring* if every subset S of k vertices is connected by a rainbow Steiner tree.

The strong k -rainbow index of a graph G is defined as the minimum number of colors required to achieve a strong k -rainbow coloring. Since every strong k -rainbow coloring is also a k -rainbow coloring, the following relationship holds:

$$rx_k(G) \leq srx_k(G).$$

Furthermore, if G is a connected graph with $|E(G)|$ edges, then the following bounds apply:

$$sdiam_k(G) \leq rx_k(G) \leq srx_k(G) \leq |E(G)|. \quad (1)$$

These concepts provide the theoretical foundation for analyzing rainbow connectivity and determining the strong k -rainbow index in various classes of graphs.

Observation 2.1. Let G be a strong 3-rainbow colored connected graph of order $n \geq 3$. If e and f are any two bridges of G then e and f are colored with distinct colors.

Theorem 2.1. For a tree T_n of order $n \geq 3$, $srx_n(T_n) = |T_n| = n - 1$

Theorem 2.2. For each integer $n \geq 3$, the strong 3 -rainbow index of C_n is

$$srx_3(C_n) = \begin{cases} 2, & \text{for } n = 3 \\ n - 3, & \text{for } 4 \leq n \leq 6 \text{ or } n = 8 \\ n, & \text{for } n = 7 \text{ or } n \geq 9 \end{cases}$$

3. Result and Discussion

Before examining the strong 3-rainbow coloring of bat graphs, we first establish the definitions of their vertices and edges.

Definition 3.1. [3] Let $n \geq 3$ be a positive integer. A Bat graph of order $2n - 1$, denoted by $Bat(n)$, is a graph with the vertex set and edge set as follows.

$$V(Bat(n)) = \{v_1, v_2, v_3, \dots, v_{2(n)-1}\} \text{ and} \\ E(Bat(n)) = \{v_i v_{i+1}, i \in [1, 2n - 2]\} \cup \{v_i v_{i+2} | i \in [1, 2n - 3]\}$$

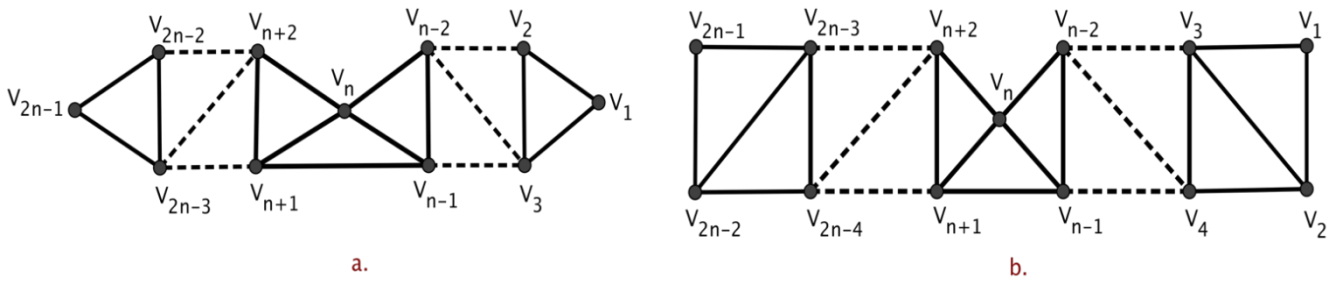


Figure 3.1. Bat (n) where (a) n is an number and (b) n is an even number

Theorem 3.1. [1] Let $n \geq 3$. Then the Steiner diameter of the bat graph $Bat(n)$ is

$$sdiam_3(Bat) = n, \text{ for } n \geq 3$$

PROOF.

Observe that for each value of n , two main situations can be considered, namely when n is odd and when n is even.

Case 1. n is odd, $n \geq 3$

In this case, all possible combinations of triple vertices in $Bat(n)$ can be grouped into four subsets, denoted by S_k for $k \in \{1, 2, 3, 4\}$, such that

$$\bigcup_{k=1}^4 S_k = S.$$

Thus, the set S represents all triple vertex combinations for odd n . The subsets are defined as follows:

$$\begin{aligned} S_1 &= \{\{v_1, v_2, v_p\} \text{ for } p = \text{odd}\} \\ S_2 &= \{\{v_1, v_2, v_q\} \text{ for } q = \text{even}\} \\ S_3 &= \{\{v_1, v_n, v_{2(n)-1}\}\} \\ S_4 &= \{\{v_2, v_n, v_{2(n)-1}\}\} \end{aligned}$$

From these cases, the Steiner distance for each subset can be determined. It follows that:

1. $sd(s \in S_1) = \frac{n+1}{2}$
2. $sd(s \in S_2) = \frac{n+1}{2}$
3. $sd(s \in S_3) = n - 1$
4. $sd(s \in S_4) = n - 1$

Case 2. n is even, $n \geq 3$,

In this situation, all triple vertex combinations can be partitioned into two subsets, namely S_5 and S_6 , whose union forms the set S . Hence, S represents all possible triple vertices for even n . These subsets are given by:

$$S_5 = \{\{v_1, v_p, v_{2(n)-1}\} \text{ for } p = \text{even}\}$$

$$S_6 = \{\{v_1, v_q, v_{2(n)-1}\} \text{ for } q = \text{odd}\}$$

The corresponding Steiner distances are obtained as follows:

1. $sd(s \in S_5) = n - 1$
2. $sd(s \in S_6) = n$

By comparing all cases, the maximum Steiner distance among all subsets is n . Therefore, the Steiner diameter of $Bat(n)$ is equal to n , which confirms the result. $\square$

Theorem 3.2. [1] *Let $n \geq 3$. The strong 3-rainbow index of the bat graph ($Bat(n)$), denoted by $srx_3(Bat(n))$ is*

$$srx_3(Bat(n)) = \begin{cases} 3, & \text{for } n = 3 \\ n + 1, & \text{for } n = 4, 5 \\ 2 \left\lfloor \frac{n}{2} \right\rfloor + 2 & \text{for } n \geq 6 \end{cases}$$

PROOF.

We prove this result by considering several cases

(Case 1.) $n = 3$,

From the general lower bound given in Inequality (1), we have

$$srx_3(Bat(3)) \geq 2.$$

To prove the upper bound, we construct an explicit edge-coloring of $Bat(3)$ using exactly 2 colors.

Let

$$c: E(Bat(3)) \rightarrow \{1, 2\}$$

be a coloring defined by assigning colors to the edges such that no two edges in any Steiner tree connecting three vertices share the same color.

$$c(v_1, v_2) = c(v_4, v_5) = 1$$

$$c(v_1, v_3) = c(v_2, v_3) = 2$$

$$c(v_3, v_4) = c(v_3, v_5) = 3$$

Under this coloring, for every subset $S \subseteq V(Bat(3))$ with $|S| = 3$, there exists a rainbow Steiner tree whose size equals $sd(S)$. Therefore, this coloring is a strong 3-rainbow coloring. Hence, we obtain

$$srx_3(Bat(3)) \leq 2.$$

Figure 3.2 which shows that every subset of three vertices is connected by a rainbow Steiner tree. ensuring that the strong 3-rainbow condition is satisfied.

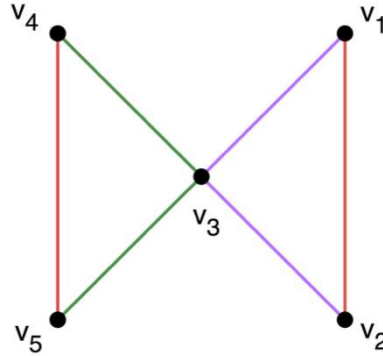


Figure 3.2. a strong 3-rainbow coloring of $Bat(3)$

Combining the lower and upper bounds, we conclude that $srx_3(Bat(3)) = 2$.

(Case 2.) $n = 4$ or $n = 5$

Since we have $Sdiam_3(Bat(n)) = n$. We have $srx_3(Bat(n)) \geq n$ by (1). Next, we have show that $srx_3(Bat(n)) \leq n+1$ by defining strong 3-rainbow coloring. $c : E(Bat(n)) \rightarrow [1, 2, \dots, n+1]$ as follows for $n = 4$ then

$$c : E(Bat(4)) \rightarrow [1, 2, 3, 4, 5]$$

$$c(v_1, v_2) = c(v_1, v_3) = 1$$

$$c(v_2, v_4) = c(v_3, v_4) = 2$$

$$c(v_4, v_5) = c(v_4, v_6) = 3$$

$$c(v_5, v_7) = c(v_6, v_7) = 4$$

$$c(v_2, v_3) = c(v_5, v_6) = 5$$

next for $n = 5$ then $c : E(Bat(5)) \rightarrow [1, 2, 3, 4, 5, 6]$

$$c(v_1, v_3) = c(v_2, v_4) = 1$$

$$c(v_3, v_5) = c(v_4, v_5) = 2$$

$$c(v_5, v_6) = c(v_5, v_7) = 3$$

$$c(v_7, v_8) = c(v_7, v_9) = 4$$

$$c(v_2, v_3) = c(v_7, v_8) = 5$$

$$c(v_1, v_2) = c(v_3, v_4) = c(v_6, v_7) = c(v_8, v_9) = 6$$

The result of the coloring is shown in figure

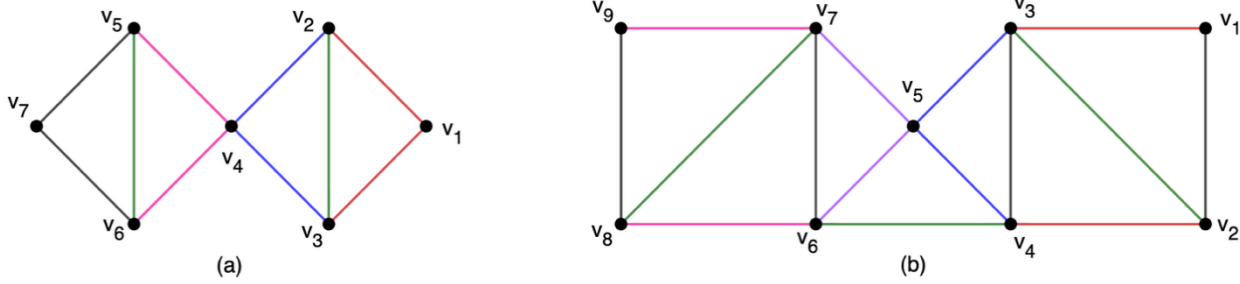


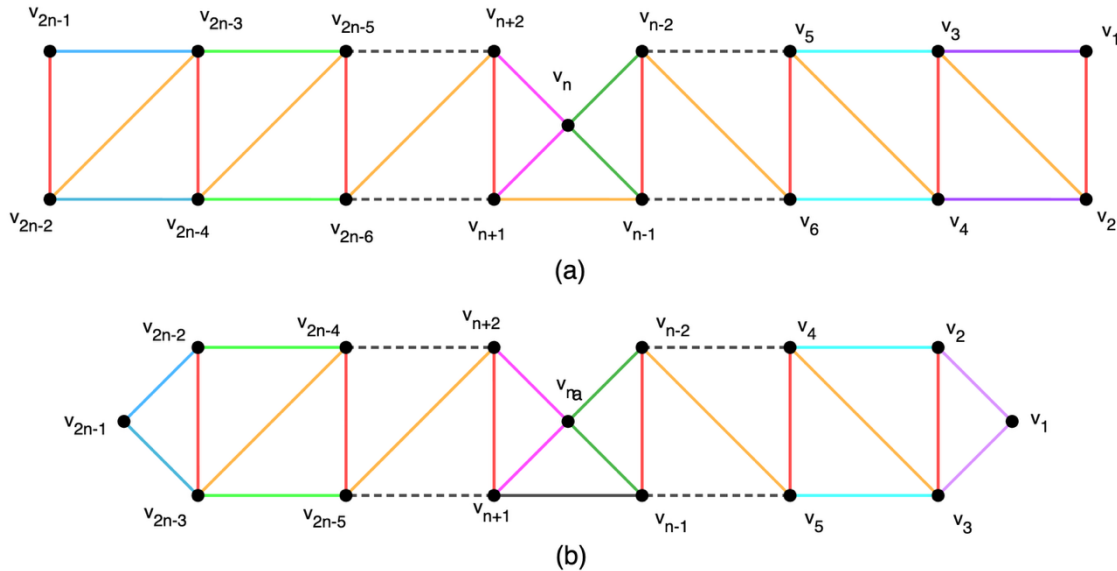
Figure 3.3. (a) a strong 3-rainbow coloring of Bat (4) and (b) a strong 3-rainbow coloring of Bat (5)

(Case 3.) $n \geq 6$

In the case, we have Since we have $Sdiam_3(Bat(n)) = n$. We have $srx_3(Bat(n)) \geq 2 \left\lfloor \frac{n}{2} \right\rfloor + 2$ by (1). Next, we have show that $srx_3(Bat(n)) \geq 2 \left\lfloor \frac{n}{2} \right\rfloor + 2$ by defining strong 3-rainbow coloring. $c : E(Bat(n)) \rightarrow [1, 2, \dots, 2 \left\lfloor \frac{n}{2} \right\rfloor + 2]$ as follows for $n \geq 6$ then $c : E(Bat(n)) \rightarrow [1, 2, 3, 4, 2 \left\lfloor \frac{n}{2} \right\rfloor + 2]$. In this case, we have two long paths. Choose the longest path, which is $\left\lfloor \frac{2n-1}{2} \right\rfloor$. Based on Theorem 2.1, the minimum number of colors required for this path is $\left\lfloor \frac{2n-1}{2} \right\rfloor - 1$. The two paths are then connected, with each vertex on each path connected to two edges, so the two edges are assigned different colors. We define a coloring of the spokes and inner rims as follows

$$\begin{aligned}
 c(v_i, v_{i+2}) &= \frac{i+1}{2}, i \in [1, 2n-3], i = \text{odd} \\
 c(v_i, v_{i+2}) &= \frac{i}{2}, i \in [2, n] \text{ and } i \in [n+1, 2n-4] \\
 c(v_i, v_{i+1}) &= n, i \in [1, n-1], i = \text{odd} \\
 c(v_i, v_{i+1}) &= n, i \in [n+1, 2n-2], i = \text{even} \\
 c(v_i, v_{i+1}) &= n+1, i \in [2, n-1], i = \text{even} \\
 c(v_i, v_{i+1}) &= n+1, i \in [n+1, 2n-3], i = \text{odd} \\
 c(v_{n-1}, v_{n+1}) &= n+1
 \end{aligned}$$

The result of the coloring is shown in figure



Gambar 3.3. a strong 3-rainbow coloring of Bat (n), (a) n is odd and (b) n is even

by observation, it is clear that $srx_3(Bat(n)) = 2 \left\lfloor \frac{n}{2} \right\rfloor + 2$, for $n \geq 6$.

Therefore, the theorem is proven. $\square$

4. Conclusion

In this paper, we investigated the strong 3-rainbow index of bat graphs $Bat(n)$ for $n \geq 3$. By applying a theoretical-analytical approach through the construction of edge colorings and structural analysis of the graph, we successfully determined the exact value of the strong 3-rainbow index for this class of graphs.

The results show that the strong 3-rainbow index of $Bat(n)$ depends on the value of n , and is given by

$$srx_3(Bat(n)) = \begin{cases} 3, & \text{for } n = 3 \\ n + 1, & \text{for } n = 4, 5 \\ 2 \left\lfloor \frac{n}{2} \right\rfloor + 2, & \text{for } n \geq 6 \end{cases}$$

These findings highlight the influence of graph structure on rainbow connectivity parameters, particularly in relation to the arrangement of paths and connectivity patterns in bat graphs. For future research, it would be interesting to extend this study to other variations of rainbow indices, such as the strong k -rainbow index for $k > 3$, or to investigate similar properties in other classes of graphs obtained through graph operations.

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