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CHARACTERIZATIONS OF DERIVATIONS INDUCED BY ENDOMORPHISMS OF BM-ALGEBRAS

KARAKTERISASI DERIVASI YANG DIINDUKSI OLEH ENDOMORFISMA DI BM-ALJABAR

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Abstract

This paper characterizes derivations associated with a fixed endomorphism f of a BM-algebra. We consider (l,r)-f-derivations, (r,l)-f-derivations, and left f-derivations. The identity $r \wedge s = r$, derived from the BM-algebra axioms, reduces their defining equations to one-sided identities. Every (l,r)-f-derivation has the form $\omega_f(r) = f(r) * (0 * c)$, where $c = \omega_f(0)$. Such a mapping is an (r,l)-f-derivation if and only if $c = 0 * c$. Left f-derivations coincide with (r,l)-f-derivations, whereas the left-right and right-left classes need not coincide. For each of the three types, regularity is equivalent to equality with f. We characterize kernel membership and prove that the kernel is a subalgebra if and only if the derivation is regular. We also characterize fixed points and prove that regular derivations have fixed-point subalgebras. Finite examples verify the definitions, distinguish the derivation classes, and show why regularity is needed in the subalgebra statements.

Keywords: BM-Algebra; f-Derivation; Endomorphism; Regularity; Kernel; Fixed-Point Set.

Abstrak

Artikel ini mengarakterisasi derivasi yang terkait dengan suatu endomorfisma tetap f pada BM-aljabar. Jenis yang dikaji meliputi (l,r)-f-derivasi, (r,l)-f-derivasi, dan f-derivasi kiri. Identitas $r \wedge s = r$, yang diturunkan dari aksioma BM-aljabar, menyederhanakan persamaan pendefinisiannya menjadi identitas satu sisi. Setiap (l,r)-f-derivasi berbentuk $\omega_f(r) = f(r) * (0 * c)$, dengan $c = \omega_f(0)$. Pemetaan tersebut merupakan (r,l)-f-derivasi jika dan hanya jika $c = 0 * c$. Kelas f-derivasi kiri berimpit dengan kelas (r,l)-f-derivasi, sedangkan kelas derivasi kiri-kanan dan kanan-kiri tidak selalu berimpit. Untuk ketiga jenis tersebut, regularitas ekuivalen dengan kesamaan terhadap f. Keanggotaan kernel dikarakterisasi dan dibuktikan bahwa kernel merupakan subaljabar jika dan hanya jika derivasinya reguler. Titik tetap juga dikarakterisasi, dan himpunan titik

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tetap derivasi reguler terbukti merupakan subaljabar. Contoh hingga diberikan untuk memverifikasi definisi, membedakan kelas derivasi, serta menunjukkan perlunya regularitas dalam pernyataan tentang subaljabar.

Kata kunci: BM-Aljabar; f-Derivasi; Endomorfisma; Regularitas; Kernel; Himpunan Titik Tetap.

INTRODUCTION

B-algebras provide a setting in which derivation identities can be formulated using a binary operation and a distinguished element. Neggers and Kim (2002) introduced B-algebras, while Kim and Kim (2006) studied BM-algebras and established their relationship with 0-commutative B-algebras. These structural identities are essential when comparing derivation notions: a formal difference between two defining equations does not, by itself, imply that they define distinct classes, and an apparent symmetry does not establish their equivalence.

Sugianti and Gemawati (2020) investigated derivations and generalized derivations of BM-algebras. Endomorphism-dependent constructions also occur in related literature. Ramadhona et al., (2020) studied generalized f-derivations of BP-algebras, and Muangkarn et al., (2021) considered f_q-derivations of B-algebras. Yattaqi et al. (2021) examined inside and outside f_q-derivations of BM-algebras. Further developments include f_q-derivations of BP-algebras (Gemawati et al., 2023) and constructions involving two endomorphisms (Muangkarn et al., 2025).

Connections with neighbouring algebraic systems provide additional structural context. Walendziak (2014) investigated BM-algebras in relation to medial BH-algebras and congruences, while Oner et al., (2022) developed Sheffer stroke BM-algebras and related structures. These studies motivate attention to the defining identities before transferring statements between algebraic settings.

For endomorphism-dependent derivations, Bantaojai et al., (2021) studied two-endomorphism formulations on B-algebras, and Bantaojai et al., (2022) considered derivations induced by left and right bi-endomorphisms. Muangkarn et al., (2022) examined outside, inside, left-right, and right-left derivations induced by an endomorphism of BG-algebras. These constructions provide context for the present use of a fixed endomorphism and for comparisons between differently ordered defining identities.

Recent work also addresses the substructures associated with derivations. Iampan et al., (2023) studied derivations of Hilbert algebras, including their kernels and fixed-point sets; Iampan et al., (2024) subsequently investigated endomorphism-dependent derivations in that setting. Phaeyai et al., (2025) examined left-right and right-left derivations of IUP-algebras, together with their kernels and fixed-point sets. These parallel questions motivate our explicit examination of the conditions under which the corresponding subsets of a BM-algebra are subalgebras.

Other developments illustrate the dependence of derivation theory on its ambient operations. Alnefaie (2024) studied reverse derivations of d-algebras, regularity, and related substructures, whereas Nakkhasen et al., (2025) investigated derivations of Bd-algebras and semigroup constructions on sets of derivations. Putri et al., (2025) introduced derivation types for pseudo BG-algebras using auxiliary operations, and Mahiroh et al., (2026) developed a derivation framework for pseudo BN-algebras. These works concern distinct defining identities; here the BM identities allow a direct classification through the value of the map at zero.

The present study works with an arbitrary self-map ω_f and a fixed endomorphism f of a BM-algebra. Rather than assuming a translated form at the outset, we determine exactly which self-maps satisfy the left-right, right-left, and left identities. The resulting characterization connects these maps to the translated constructions used in the f_q-derivation literature. Accordingly, the definitions are used within an established framework; the focus is the explicit characterization and the hypotheses needed for the structural conclusions.

The closest BM-algebra precedents are Sugianti and Gemawati (2020) and Yattaqi et al., (2021). In the former, Theorems 3.3–3.5 establish one-sided identities and characterize regular ordinary derivations by the identity map; Examples 3.1–3.2 already distinguish the two orders and exhibit a nonregular derivation. Those phenomena are therefore part of the existing theory. The present formulation replaces the identity map by a fixed endomorphism f and determines all self-maps satisfying the defining identities. Yattaqi et al., (2021) study inside and outside f_q-derivations starting from the prescribed form $f(r) * q$. Here that translated form is obtained as a necessary and sufficient characterization, with its parameter recovered uniquely from $\omega_f(0)$. This provides a common basis for the class comparison and the subsequent kernel and fixed-point criteria.

A decisive observation is that the auxiliary operation $r \wedge s = s * (s * r)$ reduces to r in every BM-algebra. We use this fact to classify the derivations, establish the precise inclusion between the classes, and prove both directions of the regularity characterization. We then give necessary and sufficient conditions for kernel membership and for the kernel to be a subalgebra. Separate propositions describe fixed points and their closure properties. All statements about subalgebras include their required hypotheses, and finite examples illustrate the limits of stronger assertions.

The research proceeds by direct algebraic deduction. We first collect the BM identities in Section 2 and prove the auxiliary identities in Lemma 2.9. With a fixed endomorphism f, we then reduce the three defining equations to (LR) and (RL). Substitution at zero and at equal arguments gives the parameter $c = \omega_f(0)$ and the necessary forms of the maps. We establish sufficiency by substituting these forms back into the defining equations, using Theorem 2.6 and the B-algebra identity (B3), which

applies by Theorem 2.5. The classification is then used to prove both directions of regularity and the criteria for kernel and fixed-point membership. Subalgebra statements are checked separately for containment of zero, closure under the binary operation, and invariance. Finally, finite examples are verified algebraically and, in Example 3.1, by checking every pair in a table.

PRELIMINARIES

Throughout the paper, R denotes the underlying set, $*$ denotes its binary operation, and 0 denotes its distinguished element. Parentheses specify the order of evaluation; associativity of $*$ is not assumed.

Definition 2.1. A B-algebra (Neggers & Kim, 2002) is a non-empty set R with a constant 0 and a binary operation $*$ satisfying, for all $r, s, t \in R$,

$$(B1)\ r * r = 0, (B2)\ r * 0 = r,$$

$$(B3)\ (r * s) * t = r * (t * (0 * s)).$$

Example 2.1. Let $R = \{0,1,2,3,4,5\}$ with the operation $*$ given in Table 1. This is the B-algebra example discussed by Kim and Kim (2006).

Table 1. Cayley Table of the Six-Element B-Algebra

*	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

For completeness, the table has the group-difference realization $r * s = rs^{-1}$ in the symmetric group on three letters. The labels $0,1,2,3,4,5$ correspond respectively to $e, (123), (132), (12), (13), (23)$, with permutation products evaluated from right to left. In any group, $r * r = e$, $r * e = r$, and both sides of (B3) equal $rs^{-1}t^{-1}$. Thus the table satisfies the B-algebra axioms.

Lemma 2.2. In a B-algebra, for all $r, s \in R$, the following standard identities hold (Neggers & Kim, 2002):

$$0 * (0 * r) = r, (r * s) * (0 * s) = r,$$

$$r * s = 0 \Leftrightarrow r = s.$$

Definition 2.3. A BM-algebra (Kim & Kim, 2006) is a non-empty set R with a constant 0 and a binary operation $*$ satisfying, for all $r, s, t \in R$,

$$(A1)\ r * 0 = r,$$

$$(A2)\ (t * r) * (t * s) = s * r.$$

Example 2.2. Let $R = \{0,1,2\}$ and define $r * s = r - s$ modulo 3. Its Cayley table is Table 2.

Table 2. Cayley Table of the Three-Element BM-Algebra

*	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

The operation satisfies $r * 0 = r$. Moreover, $(t * r) * (t * s) = (t - r) - (t - s) = s - r = s * r$ modulo 3. Hence both BM-algebra axioms hold for all choices of the variables.

Theorem 2.4. Let $(R;*,0)$ be a BM-algebra. The following identities hold for all $r, s, t \in R$ (Kim & Kim, 2006):

$$(i)\ r * r = 0;\ (ii)\ 0 * (0 * r) = r;$$
$$(iii)\ 0 * (r * s) = s * r;\ (iv)\ (r * t) * (s * t) = r * s;$$
$$(v)\ r * s = 0 \ \Leftrightarrow\ s * r = 0.$$

Theorem 2.5. Every BM-algebra is a B-algebra (Kim & Kim, 2006).

The converse fails. In Table 1, $(5 * 1) * (5 * 4) = 3 * 1 = 4$, whereas $4 * 1 = 5$. Thus (A2) fails in that B-algebra.

Theorem 2.6. If $(R;*,0)$ is a BM-algebra, then $(r * s) * t = (r * t) * s$ for all $r, s, t \in R$ (Kim & Kim, 2006).

For a BM-algebra, we use the auxiliary operation $r \wedge s = s * (s * r)$ for all $r, s \in R$. This is the same convention used in the derivation literature (Sugianti & Gemawati, 2020). The symbol does not assert that a lattice meet is present.

Definition 2.7. Let $(R;*,0)$ be a BM-algebra. A mapping $d: R \to R$ is an (l,r)-derivation if $d(r * s) = (d(r) * s) \wedge (r * d(s))$ for all $r, s \in R$. It is an (r,l)-derivation if $d(r * s) = (r * d(s)) \wedge (d(r) * s)$ for all $r, s \in R$. It is a derivation if both identities hold (Sugianti & Gemawati, 2020).

Definition 2.8. A mapping $d: R \to R$ is regular if $d(0) = 0$. An endomorphism $f: R \to R$ satisfies $f(r * s) = f(r) * f(s)$ for all $r, s \in R$. Every such endomorphism preserves 0, since $f(0) = f(r * r) = f(r) * f(r) = 0$ by Theorem 2.4 (i).

Lemma 2.9. Let $(R;*,0)$ be a BM-algebra. For all $r, s \in R$,

$$(i)\ r \wedge s = r;\ (ii)\ r * s = 0 \ \Leftrightarrow\ r = s;$$
$$(iii)\ r * (0 * s) = s * (0 * r);\ (iv)\ (r * s) * (0 * s) = r.$$

Proof. For (i), use (A2) with $t = s$, its first subtracted argument 0, and its second subtracted argument r. By (A1),

$$r \wedge s = s * (s * r) = (s * 0) * (s * r) = r * 0 = r.$$

For (ii), if $r * s = 0$, then (i) gives $s = r * (r * s) = r * 0 = r$. The converse is Theorem 2.4 (i). For (iii), use (A2) with $t = 0$ and Theorem 2.4 (ii):

$$r * (0 * s) = (0 * (0 * r)) * (0 * s) = s * (0 * r).$$

Finally, Theorem 2.5 permits the use of (B3), so

$$(r * s) * (0 * s) = r * ((0 * s) * (0 * s)) = r * 0 = r.$$

This proves (iv). All four statements apply directly to the BM-algebra under consideration.□

RESEARCH RESULTS AND DISCUSSION

Throughout this section, $(R;*,0)$ is a BM-algebra and $f: R \to R$ is a fixed endomorphism. The map $\omega_f: R \to R$ is not initially assumed to preserve $*$ or 0.

Definition 3.1. For all $r, s \in R$, a map $\omega_f: R \to R$ is an (l, r)-f-derivation if

$$\omega_f(r * s) = (\omega_f(r) * f(s)) \wedge (f(r) * \omega_f(s)),$$

and it is an (r, l)-f-derivation if

$$\omega_f(r * s) = (f(r) * \omega_f(s)) \wedge (\omega_f(r) * f(s)).$$

A map satisfying both identities is an f-derivation. We call ω_f a left f-derivation if

$$\omega_f(r * s) = (f(r) * \omega_f(s)) \wedge (f(s) * \omega_f(r)).$$

The first two definitions follow the established endomorphism-dependent formulation (Muangkarn et al., 2021). The adjective left refers specifically to the last displayed identity. Regularity means $\omega_f(0) = 0$. By Lemma 2.9 (i), these definitions reduce respectively to

$$(LR)\ \omega_f(r * s) = \omega_f(r) * f(s),$$
$$(RL)\ \omega_f(r * s) = f(r) * \omega_f(s).$$

The left identity also reduces to (RL). Thus left f-derivations and (r, l)-f-derivations coincide. This observation will be used explicitly in the proofs.

Example 3.1. Let $R = \{0, a, b\}$ with the operation $*$ given in Table 3, and define $f(0) = 0$, $f(a) = b$, $f(b) = a$, and $\omega_f = f$.

Table 3. Cayley Table for the Example

*	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

Identify a with 1 and b with 2 in $\mathbb{Z}_3$. Then $r * s = r - s$ modulo 3, so the verification in Example 2.2 shows that this is a BM-algebra. The map is $f(r) = -r$ modulo 3, and therefore

$$f(r * s) = -(r - s) = (-r) - (-s) = f(r) * f(s).$$

This proves that f is an endomorphism for every pair. Table 4 gives a direct check of all nine pairs. Each displayed entry is the common value of $f(r * s)$, $f(r) * f(s)$, $\omega_f(r * s)$, $\omega_f(r) * f(s)$, and $f(r) * \omega_f(s)$.

Table 4. Cayley Table for the Example

r \ s	0	a	b
0	0	a	b
a	b	0	a
b	a	b	0

Because $u \wedge v = u$ in this BM-algebra, the same table verifies both wedge identities and the left identity in Definition 3.1. Consequently, ω_f is a regular f-derivation and a left f-derivation.

Characterization and Regularity

Evaluating the left-right identity at equal arguments gives a useful relation between the value at zero and the endomorphism.

Lemma 3.2. If ω_f is an (l, r)-f-derivation, then $\omega_f(0) = \omega_f(r) * f(r)$ for every $r \in R$.

Proof. By Theorem 2.4 (i) and (LR),

$$\omega_f(0) = \omega_f(r * r) = \omega_f(r) * f(r).$$

This holds for every $r \in R$.□

Corollary 3.3. An (l, r)-f-derivation ω_f is regular if and only if $\omega_f = f$.

Proof. If $\omega_f(0) = 0$, Lemma 3.2 yields $\omega_f(r) * f(r) = 0$ for every $r \in R$. Lemma 2.9 (ii) gives $\omega_f(r) = f(r)$, hence $\omega_f = f$. Conversely, if $\omega_f = f$, then $\omega_f(0) = f(0) = 0$ by Definition 2.8.□

Theorem 3.4. A mapping $\omega_f: R \to R$ is an (l, r)-f-derivation if and only if there exists $c \in R$ such that

$$\omega_f(r) = f(r) * (0 * c) \; for \; all \; r \in R.$$

The parameter is unique and equals $c = \omega_f(0)$.

Proof. Suppose first that ω_f satisfies (LR), and set $c = \omega_f(0)$. Theorem 2.4 (ii), (LR), and the endomorphism property give

$$\omega_f(r) = \omega_f(0 * (0 * r)) = c * f(0 * r) = c * (0 * f(r)).$$

By Lemma 2.9 (iii), $c * (0 * f(r)) = f(r) * (0 * c)$, proving the required formula. Conversely, fix $c \in R$ and define $\omega_f(r) = f(r) * (0 * c)$. Theorem 2.6 gives

$$\omega_f(r * s) = (f(r) * f(s)) * (0 * c)$$
$$= (f(r) * (0 * c)) * f(s) = \omega_f(r) * f(s).$$

Thus (LR) holds. Finally, $\omega_f(0) = 0 * (0 * c) = c$ by Theorem 2.4 (ii), so the parameter is unique.□

The formula in Theorem 3.4 has the translated form $f(r) * q$ with $q = 0 * c$, as used in the f_q-derivation framework of Muangkarn et al., (2021) and Yattaqi et al. (2021). Because $0 * (0 * c) = c$, this change of parameter is reversible. Thus the characterization recovers the translated family from an arbitrary self-map and identifies its unique parameter through the value at zero. The inside identity is (LR), while the outside identity is (RL); Theorem 3.5 specifies exactly when a member of this family satisfies the latter.

Theorem 3.5. A mapping $\omega_f: R \to R$ is an (r, l)-f-derivation if and only if there exists $c \in R$ such that $c = 0 * c$ and

$$\omega_f(r) = f(r) * c \; for \; all \; r \in R.$$

Here $c = \omega_f(0)$. Moreover, $\omega_f(0) = f(r) * \omega_f(r)$ for every $r \in R$.

Proof. Suppose (RL) holds and set $c = \omega_f(0)$. Substituting $s = 0$ gives $\omega_f(r) = f(r) * c$. Substituting $r = s = 0$ gives $c = 0 * c$. Conversely, suppose these two conditions hold. By (B3), valid by Theorem 2.5, and by Theorem 2.4 (ii),

$$f(r) * (f(s) * c) = (f(r) * (0 * c)) * f(s).$$

Since $0 * c = c$, Theorem 2.6 now gives

$$f(r) * \omega_f(s) = (f(r) * c) * f(s)$$
$$= (f(r) * f(s)) * c = \omega_f(r * s).$$

Hence (RL) holds. Also $\omega_f(0) = 0 * c = c$. Finally, apply (RL) to $r * r = 0$ to obtain $\omega_f(0) = f(r) * \omega_f(r)$.□

Corollary 3.6. An (r, l)-f-derivation, or equivalently a left f-derivation, is regular if and only if it equals f.

Proof. For an (r, l)-f-derivation, Theorem 3.5 gives $\omega_f(r) = f(r) * \omega_f(0)$. If it is regular, (A1) yields $\omega_f(r) = f(r)$. Conversely, equality with f implies regularity because $f(0) = 0$. The left case follows from the reduction of its identity to (RL).□

Theorem 3.7. Every (r, l)-f-derivation is an (l, r)-f-derivation. If ω_f is an (l, r)-f-derivation and $c = \omega_f(0)$, then the following are equivalent: it is an (r, l)-f-derivation; it is a left f-derivation; it is an f-derivation; and $c = 0 * c$.

Proof. By Theorem 3.5, an (r,l)-f-derivation has the form $f(r) * c$ with $c = 0 * c$. This equals $f(r) * (0 * c)$, so Theorem 3.4 proves that it is an (l,r)-f-derivation. For a given (l,r)-f-derivation, Theorems 3.4 and 3.5 show that it also satisfies (RL) exactly when $c = 0 * c$. The equivalence with the left condition follows from Definition 3.1, and the equivalence with the f-derivation condition follows from the definition of that term.□

In particular, all three regular classes consist of the single mapping f. On the other hand, the unrestricted left-right and right-left classes coincide for a fixed f if and only if $0 * c = c$ for every $c \in R$: sufficiency follows from Theorem 3.7, while necessity follows by applying it to each map furnished by Theorem 3.4.

Example 3.2. The converse inclusion in Theorem 3.7 fails in general. On $R = \mathbb{Z}_3$ with $r * s = r - s$ modulo 3, take $f(r) = r$ and $\omega_f(r) = r + 1$ modulo 3.

Here $c = 1$ and $0 * c = 2$, so Theorem 3.4 applies but $c = 0 * c$ fails. Explicitly, $\omega_f(r * s) = r - s + 1 = \omega_f(r) * f(s)$ for all $r, s \in R$. In contrast, at $r = s = 0$ we have $\omega_f(0 * 0) = 1$ and $f(0) * \omega_f(0) = 2$. Thus ω_f is an (l,r)-f-derivation but is neither an (r,l)-f-derivation nor a left f-derivation.

Example 3.3. Nonregular f-derivations can also occur. On $R = \mathbb{Z}_4$ with $r * s = r - s$ modulo 4, let $f(r) = r$ and $\omega_f(r) = r + 2$ modulo 4.

In this case $c = 2 = 0 * 2$. Therefore Theorems 3.4–3.7 show that ω_f satisfies all three identities. It is not regular, since $\omega_f(0) = 2$. Directly, both $\omega_f(r) * f(s)$ and $f(r) * \omega_f(s)$ equal $r - s + 2$ modulo 4. This example shows that being an f-derivation does not imply regularity.

Kernels and Fixed Points

Definition 3.8. For a mapping $\omega_f : R \to R$, define its kernel and fixed-point set by

$$ker(\omega_f) = \{r \in R : \omega_f(r) = 0\},$$
$$Fix(\omega_f) = \{r \in R : \omega_f(r) = r\}.$$

The word kernel denotes the zero preimage; it does not assume that ω_f is a homomorphism. A subalgebra of $(R;*,0)$ is a subset containing 0 and closed under $*$. A subset is invariant under a map if the map sends every element of that subset into the same subset.

Theorem 3.9. Let ω_f be an (l,r)-f-derivation and let $c = \omega_f(0)$. Then, for every $r \in R$,

$$r \in ker(\omega_f) \Leftrightarrow f(r) = 0 * c.$$

Consequently, the kernel is non-empty if and only if $0 * c \in f(R)$. Furthermore, $ker(\omega_f)$ is a subalgebra if and only if ω_f is regular. In that case $ker(\omega_f) = ker(f)$, and this subalgebra is invariant under both f and ω_f. These conclusions also apply to (r,l)-f-derivations and left f-derivations.

Proof. By Theorem 3.4 and Lemma 2.9 (ii),

$$\omega_f(r) = 0 \Leftrightarrow f(r) * (0 * c) = 0 \Leftrightarrow f(r) = 0 * c.$$

This proves the membership criterion and the non-emptiness criterion. If the kernel is a subalgebra, it contains 0, so $\omega_f(0) = 0$. Conversely, if ω_f is regular, Corollary 3.3 gives $\omega_f = f$. Therefore $0 \in ker(\omega_f)$, and for $r, s \in ker(\omega_f)$,

$$\omega_f(r * s) = f(r * s) = f(r) * f(s) = 0 * 0 = 0.$$

Thus the kernel is closed under $*$. If $r \in ker(f)$, then $f(r) = 0 \in ker(f)$, so it is invariant under f and, because $\omega_f = f$, under ω_f. Theorem 3.7 and the equality of the left and right-left classes extend the conclusions to the other types.□

The regularity hypothesis cannot be omitted. In Example 3.2, $ker(\omega_f) = \{2\}$, and $2 * 2 = 0 \notin ker(\omega_f)$. In Example 3.3 the kernel is again $\{2\}$ and fails the same closure test, even though the map satisfies all three derivation identities.

Corollary 3.10. Let ω_f be a regular derivation of any of the three types in Definition 3.1. Then $Fix(\omega_f) = Fix(f)$ is a subalgebra of R. It is invariant under f and ω_f.

Proof. Corollaries 3.3 and 3.6 give $\omega_f = f$. Since $f(0) = 0$, the fixed-point set contains 0. If $r, s \in Fix(f)$, then

$$f(r * s) = f(r) * f(s) = r * s,$$

so it is closed under $*$. Each fixed point is sent to itself by f and ω_f, proving invariance.□

Proposition 3.11. Let ω_f be an (l, r)-f-derivation and let $c = \omega_f(0)$. Then

$$r \in Fix(\omega_f) \Leftrightarrow f(r) = r * c.$$

The set $Fix(\omega_f)$ is a subalgebra if and only if ω_f is regular. If f is the identity map, the fixed-point set is R when $c = 0$ and is empty when $c \neq 0$.

Proof. Theorem 3.4 gives $r \in Fix(\omega_f)$ exactly when $f(r) * (0 * c) = r$. Applying the operation $* c$ to this equality and using Lemma 2.9 (iv) together with Theorem 2.4 (ii) gives $f(r) = r * c$. Conversely, $f(r) = r * c$ implies

$$\omega_f(r) = (r * c) * (0 * c) = r$$

by Lemma 2.9 (iv). If the fixed-point set is a subalgebra, then 0 belongs to it, so $\omega_f(0) = 0$. The converse follows from Corollary 3.10.

Finally, suppose f is the identity. If $c = 0$, Corollary 3.3 yields $\omega_f = f$, so every element is fixed. If a fixed point r existed when $c \neq 0$, the membership criterion would give $r = r * c$. But Lemma 2.9 (i) implies $r * (r * c) = c$, and hence $c = r * r = 0$, a contradiction. Thus the fixed-point set is empty when $c \neq 0$.□

Remark. Two elementary observations hold independently of regularity. Since $r \wedge s = r$, every subset is closed under $\wedge$. Also, any self-map g satisfies $g(g(r)) = r$ at each of its fixed points. Neither observation establishes closure under $*$ or idempotence on the whole algebra.

Example 3.4. Let $R = \mathbb{Z}_6$ with $r * s = r - s$ modulo 6, and define $f(r) = 4r$ modulo 6 and $\omega_f = f$.

As in Example 2.2, subtraction modulo 6 satisfies (A1) and (A2). Moreover, $f(r * s) = 4(r - s) = 4r - 4s = f(r) * f(s)$ modulo 6, so f is an endomorphism. By Theorems 3.4 and 3.5 with $c = 0$, the mapping ω_f is a regular derivation of all three types. Its kernel and fixed-point set are

$$ker(\omega_f) = \{0,3\}, \qquad Fix(\omega_f) = \{0,2,4\}.$$

Indeed, $4r = 0$ modulo 6 exactly for $r = 0,3$, and $4r = r$ modulo 6 exactly for $r = 0,2,4$. Both sets contain 0 and are closed under subtraction modulo 6. This gives explicit proper subalgebras illustrating Theorem 3.9 and Corollary 3.10.

Interpretation of the Characterizations

The parameter $c = \omega_f(0)$ controls the departure of a derivation from the endomorphism f. The left-right identity permits every $c \in R$, whereas the right-left and left identities require $c = 0 * c$. Regularity imposes the stronger condition $c = 0$. Accordingly, the common regular class contains only f, but the common unrestricted class may also contain nonregular maps, as Example 3.3 shows.

The descriptions of kernels and fixed points follow from the same parameter. Kernel membership is the equation $f(r) = 0 * c$, while the fixed-point condition is $f(r) = r * c$. Requiring either set to be a subalgebra forces $c = 0$, because a subalgebra must contain the distinguished element. In that case the sets reduce to the usual kernel and fixed-point subalgebras of the endomorphism. These statements specify the structural content of the results without asserting an unconditional equivalence of the derivation classes.

The kernel and fixed-point results address questions also considered for Hilbert and IUP-algebras (Iampan et al., 2023, 2024; Phaeyai et al., 2025). The BM-specific classification here supplies an explicit criterion: regularity is necessary and sufficient for either set to be a subalgebra. This conclusion follows from the translated form and the requirement that a subalgebra contain zero; it does not rely on transferring a closure result from another algebraic system.

For f equal to the identity, Corollaries 3.3 and 3.6 recover the regularity characterization of Sugianti and Gemawati (2020). The examples distinguishing the orders illustrate this established phenomenon in finite algebras. The additional structural conclusions here concern a fixed, possibly nonidentity endomorphism: Theorem 3.9 and Proposition 3.11 give explicit membership equations and show that regularity is necessary and sufficient for the corresponding zero preimage or fixed-point set to

be a subalgebra. Example 3.4 illustrates the regular case with a nonidentity endomorphism and two proper subalgebras.

CONCLUSION AND SUGGESTIONS

For a fixed endomorphism f of a BM-algebra, every (l,r)-f-derivation is uniquely determined by its value c at 0 and has the form $\omega_f(r) = f(r) * (0 * c)$. It is an (r,l)-f-derivation precisely when $c = 0 * c$. Left f-derivations coincide with (r,l)-f-derivations, and each is an (l,r)-f-derivation. The reverse inclusion fails in general. For each of these types, regularity holds if and only if $\omega_f = f$.

The kernel is characterized by $f(r) = 0 * c$, and the fixed-point set by $f(r) = r * c$. Each is a subalgebra if and only if the derivation is regular. In the regular case, they are respectively the kernel and fixed-point subalgebras of f, and both are invariant under f and ω_f. The finite examples demonstrate both the validity of these results and the necessity of distinguishing unrestricted derivations from regular ones.

Further work may examine identities involving two endomorphisms or analogous classifications in structures where the auxiliary operation ∧ does not reduce to a projection. Such extensions should first determine the relevant structural identities, since these identities decide whether apparently different derivation notions actually coincide. The present results are algebraic; applications to logic or computation would require a separate construction and analysis.

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